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Regulated Extensions in Rational Chebyshev Approximation

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Let $V(N, \mu) \subset \mathbf{C}(X)$ be a set of rationals of the form B/L^μ with $B, L \in \mathbf{C}(X)$, $L(x) > 0\ \forall x \in X$, and $\mu \in \mathbf{N}$; we study existence of best approximations for extensions of $V(N, \mu)$ into the space of regulated functions $R(X)$. We show existence of best approximations for an extension $V_0(N, \mu)$ which is maximal in the sense that the quality of approximation on $V_0(N, \mu)$ is the best we can achieve by any proper rational extension of $V(N, \mu)$.

Then we consider the problem of characterization of minimal extensions of $V(N, \mu)$ which possess stabilized best approximations. We show that a quasi-minimal extension $V^0(N, \mu)$ similar to that defined by Rice [4, p. 82] in general is no minimal extension and give a necessary and sufficient condition for $V^0(N, 1)$ being minimal.

Finally we want to give some sufficient conditions under which $V^0(N, \mu)$ is a minimal extension for large enough $\mu \in \mathbf{N}$.

1. Regulated Functions

Let $X \subset \mathbf{R}^n$ be a compact connected subset of $\mathbf{R}^n$ and consider the class $R(X)$ of *regulated* functions $f: X \to \mathbf{R}$ satisfying:

(i) f is continuous $\forall x \notin X_f$, where X_f is l.c. in X.

(ii) $|f(x)| \leqslant M < \infty\ \forall x \notin X_f$.

(iii) $f(x) = \frac{1}{2}(m_f(x) + M_f(x))\ x \in X_f$.

For a definition of *first category* (l.c.) sets X_f in X see Kantorowitsch–Akilow [2, p. 23], and the numbers $m_f(x)$ and $M_f(x)$ for $x \in X_f$ are defined by

$$m_f(x) = \inf(\liminf_{j\to\infty} f(x_j)),$$
$$M_f(x) = \sup(\limsup_{j\to\infty} f(x_j)) \qquad x \in X_f,$$

where the infimum and the supremum are taken over all sequences $(x_j) \not\subset X_f$, $x_j \to x \in X_f$.

Condition (iii) above should be understood in the sense that although we assign fixed values to f on X_f we still may have multivalued convergence:

$$f(y) \to [m_f(x), M_f(x)] \qquad \text{for} \quad y \to x \in X_f.$$

Examples for regulated functions are $f(x) = \sin(\pi/x) \in R(I)$ with $f(0) = 0$ and $f(y) \to I$ for $y \to 0$, and in two dimensions

$$f(x) = u^2/(u^2 + v^2) \in R(I \times I),$$

$x = (u, v)' \in \mathbf{R}^2$ and $f(0) = \frac{1}{2}$ and $f(y) \to [0, 1]$ $y \to 0$, with $I = [-1, +1]$ in both examples.

Introducing condition (iii) rather than dealing with the multivalues allows us to define addition and scalar multiplication on $R(X)$ in the usual way; it is straightforward to show that

$$(f + g)(x) = \tfrac{1}{2}(m_{f+g}(x) + M_{f+g}(x)) = f(x) + g(x), \qquad x \in X_f \vee X_\lambda,$$
$$(\beta f)(x) = \tfrac{1}{2}(m_{\beta f}(x) + M_{\beta f}(x)) = \beta f(x), \qquad x \in X_f,$$

which then gives the following result:

LEMMA. *Let $f, g \in R(X)$. Then we have*

$$\sup_{x \notin X_f \vee X_g} |f(x) - g(x)| = \sup_{x \in X} |f(x) - g(x)|.$$

The proof offers no further insight and is therefore omitted; an important result however of this lemma is that $R(X)$ is a normed space with the norm

$$\|f\| = \sup_{x \notin X_f} |f(x)|.$$

2. Regulated Extensions of Rationals

Let $S, T \subset \mathbf{C}(X)$ be two finite-dimensional subspaces of $\mathbf{C}(X)$ with normalized bases $\{u_1, ..., u_k\}$ and $\{v_1, ..., v_l\}$ and put $N = k + l$. With

$$B_a = \sum_{i=1}^{k} a_i u_i, \qquad L_b = \sum_{i=1}^{l} b_i v_i,$$

the classical problem then is to find a best rational approximation to $f \in R(X)$ on the set

$$V(N, \mu) = F_\mu(P) \subset \mathbf{C}(X),$$

with the mapping

$$F_\mu : \mathbf{R}^N \to \mathbf{C}(X), \qquad F_{\mu,c} = B_a/L_b{}^\mu \qquad c \in P \subset \mathbf{R}^N, \qquad \mu \in \mathbf{N}$$

and the parameter set

$$P = \{c = (a, b)' \in \mathbf{R}^N \mid a \in \mathbf{R}^k; \quad b \in \mathbf{R}^l, \quad \| b \| = 1, \quad L_b(x) > 0 \quad \forall x \in X\},$$

and it is well known that $V(N, \mu)$ in general is no existence set even if we restrict $f \in \mathbf{C}(X)$.

Hence we want to consider extensions of F_μ mapping certain extensions of P into $R(X)$. To do this we need the following basic *assumption*: *Any function* $L_b \in T$, $\| b \| = 1$ *has at most a* l.c. *zero-set* X_b *in* X. Furthermore we make the *assumption* $P \neq \varnothing$ to avoid consideration of trivial cases.

We first consider the following two extensions of P:

$$P_* = \{c \in \mathbf{R}^N \mid a \in \mathbf{R}^k; b \in \mathbf{R}^l, \| b \| = 1\},$$
$$P_0 = \{c \in P_* \mid F_{\mu,c} \in R(X)\},$$

and define the numbers

$$M_* = \inf\{\beta \mid \exists c \in P_* : | B_a(x) - f(x) L_b{}^\mu(x)| \leqslant \beta \mid L_b(x)|^\mu \ \forall x \in X,$$
$$M_0 = \inf\{\beta \mid \exists c \in P_0 : \| F_{\mu,c} - f \| \leqslant \beta\}.$$

Clearly we have $0 \leqslant M_* \leqslant M_0$ and we can generalize the lemma in Section 4 of Goldstein [1].

LEMMA. *$M_* = M_0$; $M_* > 0$ if and only if $f \notin V_0(N, \mu) = F_\mu(P_0)$; $c \in P_*$ and $| B_a(x) - f(x) L_b{}^\mu(x)| \leqslant M_* \mid L_b(x)|^\mu \ \forall x \in X$ are consistent; and $c \in P_0$ and $\| F_{\mu,c} - f \| \leqslant M_0$ are consistent.*

The proof is essentially that of Goldstein [1] and hence omitted. It should be noted that both M_* and M_0 depend on the fixed exponent $\mu \in \mathbf{N}$ chosen, and we can interprete the result in the sense that for a fixed $\mu \in \mathbf{N}$ the best quality of approximation we can expect by any proper extension of $V(N, \mu)$ is M_*, and that this quality actually is achieved on the set $V_0(N, \mu) \subset R(X)$. Thus we want to call $V_0(N, \mu)$ a maximal extension of $V(N, \mu)$ and note that this property does not depend on the specific function $f \in R(X)$ we wish to approximate.

To give a simple example consider the Heaviside function $H \in R(I)$:

$$H(x) = \begin{cases} +1 & \text{for } x > 0, \\ \frac{1}{2} & \text{for } x = 0, \\ 0 & \text{for } x < 0, \end{cases} \qquad x \in I = [-1, +1],$$

and rationals of the form

$$F_{1,c} = (a_1 x + a_2 \mid x \mid)/(b_1 + b_2 x) \in R(I),$$

where we have $M_0 = M_* = 0$.

Next we want to study minimal extensions of $V(N, \mu)$ which do not improve on the quality

$$M = \inf\{\beta \mid \exists c \in P : \| F_{\mu,c} - f \| \leqslant \beta\}$$

of approximation on $V(N, \mu)$, but do guarantee the existence of a best approximation. It is clear from Goldstein's terminology that only minimal extensions of $V(N, \mu)$ possess stabilized best approximations. Similar to Rice's approach in [4] we define the parameter sets

$$P^* = \{c \in P_* \mid L_b(x) \geqslant 0, \forall x \in X\},$$

$$P^0 = \{c \in P^* \mid \exists (c_j) \subset P: c_j \to c, \| F_{\mu,c_j} \| \leqslant K < \infty\},$$

which then give the following result.

LEMMA. *Let $c \in P_0$ with $L_b(x) \geqslant 0 \quad \forall x \in X$. Then $c \in P^0$.*

Proof. Since $P \neq \varnothing$ there exists a vector $c_1 \in P$ such that

$$L_{b_1}(x) > 0 \qquad \forall x \in X$$

and for $\epsilon > 0$
we define

$$c_\epsilon = (a, \{b + \epsilon b_1\}/\| b + \epsilon b_1 \|) \in P$$

for sufficiently small $\epsilon > 0$. Clearly $c_\epsilon \to c$ for $\epsilon \to 0$, and furthermore

$$| F_{\mu,c_\epsilon}(x)| = | B_a(x)|/L^\mu_{b_\epsilon}(x) \leqslant \| b + \epsilon b_1 \| \cdot | F_{\mu,c}(x)| \leqslant K < \infty \qquad \forall \epsilon \in (0, \epsilon_0).$$

Defining the constants

$$M^0 = \inf\{\beta \mid \exists c \in P^0 : \| F_{\mu,c} - f \| \leqslant \beta\},$$

$$M^* = \sup\{\beta \mid \nexists c \in P^* : | B_a(x) - f(x) L_b{}^\mu(x)| < \beta L_b{}^\mu(x) \quad \forall x \notin X_f\},$$

it follows with the above lemma that $M^0 \leqslant M^* \leqslant M$, for if $\epsilon > 0$ arbitrary we have a vector $c \in P^*$ such that

$$| B_a(x) - f(x) L_b{}^\mu(x)| < (M^* + \epsilon) L_b{}^\mu(x) \quad \forall x \notin X_f$$

and thus $c \in P^0$ and $\| F_{\mu,c} - f \| \leqslant M^* + \epsilon$.

$V^0(N, \mu) = F_\mu(P^0) \subset R(x)$ is quasi minimal in the sense that any set

$$P_K{}^0 = \{c \in P^0 \mid \| F_{\mu,c} \| \leqslant K\} \subset P^0$$

is compact in $\mathbf{R}^N$, and P is dense in P^0. Furthermore any minimal parameter sequence $(c_j) \subset P$ contains a subsequence converging to a vector $c \in P^0$ with

$\|F_{\mu,c} - f\| \leq M$, for details see [5]. $V^0(N, \mu)$ itself is an existence set, the proof is by Goldstein [1]: Let $M_j \to M^0$ and define

$$W_j = \{c \in P^0 \mid \|F_{\mu,c} - f\| \leq M_j\},$$

then all W_j are compact, $W_{j+1} \subset W_j$ and $W_1 \neq \varnothing$. Thus $\bigcap_j W_j \neq \varnothing$.

However, $V^0(N, \mu)$ in general is no minimal extension of $V(N, \mu)$, as Goldstein has shown in [1] for the case $\mu = 1$, and we want to add a different characterization of minimal extensions.

THEOREM. *Let $f \in R(X)$ and $\mu = 1$. Then $M^0 = M$ if and only if for all $\epsilon > 0$ there exists a pair $c_1, c_2 \in P^0$ with*

$$\|F_{1,c_i} - f\| \leq M^0 + \epsilon \qquad i = 1, 2 \quad \textit{and} \quad X_{b_1} \cap X_{b_2} = \varnothing.$$

Proof. ($\Rightarrow$) this directions is trivially true, since $\forall \epsilon > 0\ \exists c \in P$ with $\|F_{1,c} - f\| \leq M^0 + \epsilon$ and $X_b = \varnothing$. ($\Leftarrow$) Suppose $\exists c_1, c_2 \in P^0$ such that $X_{b_1} \cap X_{b_2} = \varnothing$ and $\|F_{1,c_i} - f\| \leq M^0 + \epsilon$, $i = 1, 2$ and $\epsilon > 0$ arbitrary fixed. Then define

$$c_\lambda = (\{1 - \lambda\} c_1 + \lambda c_2)/t_\lambda,$$

with $\lambda \in (0, 1)$ such that $t_\lambda = \|(1 - \lambda) b_1 + \lambda b_2\| > 0$. Clearly $c_\lambda \in P$ and

$$\begin{aligned} &|B_{a_\lambda}(x) - f(x) L_{b_\lambda}(x)| \\ &\quad \leq (1 - \lambda)/t_\lambda \cdot |B_{a_1}(x) - f(x) L_{b_1}(x)| + \lambda/t_\lambda \cdot |B_{a_2}(x) - f(x) L_{b_2}(x)| \\ &\quad \leq \{M^0 + \epsilon\} \cdot L_{b_\lambda}(x) \qquad \forall x \in X, \end{aligned}$$

and hence $M \leq \|F_{1,c} - f\| \leq M^0$.

COROLLARY. *Let $f \in R(X)$ and $\mu = 1$. Then $M^0 = M^*$ if and only if for all $\epsilon > 0$ there exists a pair $c_1, c_2 \in P^0$ with*

$$\|F_{1,c_i} - f\| \leq M^0 + \epsilon \qquad i = 1, 2 \quad \textit{and} \quad X_{b_1} \cap X_{b_2} \subset X_f.$$

$M^* = M$ if and only if for all $\epsilon > 0$ there exists a vector $c \in P^0$ such that $\|F_{1,c} - f\| \leq M^* + \epsilon$ and $X_b \cap X_f = \varnothing$.

The proof follows as in the above theorem. We want to give two examples. First consider approximation of $f(x) = \sin(\pi x)/x \in \mathbf{C}(I)$, $I = [-1, +1]$ by functions

$$F_{1,c}(x) = a \cdot \sin(\pi \cdot |x|)/(b_1 + b_2 |x|) \in R(I).$$

Here we have $0 = M^0 < M^* = M = 1$. Obviously $M^* = M$ for any continuous function $f \in \mathbf{C}(X)$; in the general case $f \in R(X)$ this may not be so, as for approximation of the Heaviside function $H(x)$ by regulated rationals

$$F_{1,c}(x) = (a_1 x + a_2 \mid x \mid)/(b_1 + b_2 \mid x \mid) \in R(I),$$

where we have $0 = M^0 < M^* < M = 1$.

3. Sufficient Conditions for $M^0 = M$

The above theorems do not generalize to the case $\mu > 1$ since $V^0(N, \mu)$ is not asymptotically convex. Therefore we want to give some sufficient conditions which guarantee $M^0 = M$ for sufficiently large $\mu \in \mathbf{N}$, assuming additional knowledge about the sets S, $T \subset C(X)$.

The set $S < \mathbf{C}(X)$ satisfies *condition* (Z) with some constant $\xi < \infty$, if for any $B_a \in S$, $\| a \| = 1$ with a zero at $x \in X$ there exists a sequence $x_j \to x$ such that:

$$| B_a(x_j) | \geqslant \rho \cdot \| x_j - x \|^{\xi} \qquad \forall j > N(x, a)$$

and $\rho > 0$ independent of a and $x \in X$. Then we have the following result.

THEOREM. *Suppose the set $S < \mathbf{C}(X)$ satisfies condition (Z) for some constant $\xi < \infty$ and $T < \mathbf{C}(X)$ is a Hölder set with Hölder constant $\lambda > 0$. Then for $\mu > \xi/\lambda$ we have $M^0 = M$.*

Proof. To recall the definition of Hölder sets $T < \mathbf{C}(X)$, we have the existence of a number $\lambda > 0$ such that:

$$| L_b(x) - L_b(y) | \leqslant \sigma \cdot \| x - y \|^{\lambda} \qquad \forall x, y \in X$$

and for all functions $L_b \in T$, $\| b \| = 1$, where again $\sigma > 0$ is independent of b.

Now let $c \in P^0$ such that $\| F_{\mu,c} - f \| = M^0$. Suppose $c = (0, b)'$ then nothing has to be proved, since in this case we have $F_{\mu,c}(x) = 0 \quad \forall x \notin X_b$, and thus $M^0 = \| f \|$. But $M \leqslant \| f \|$ since $P \neq \varnothing$.

Thus assume $c = (a, b)'$ with $\| a \| > 0$. Then if $X_b \neq \varnothing$ we have $B_a(x) = 0$ for any $x \in X_b$ and furthermore a sequence $x_j \to x \in X_b$ such that

$$\| F_{\mu,c} \| \geqslant | B_a(x_j) | / L_b(x_j) \geqslant \rho / (\| a \| \cdot \sigma) \cdot \| x - x_j \|^{\xi - \lambda \cdot \mu}$$
$$\to \infty \qquad \text{for} \quad x_j \to x,$$

which is impossible, i.e., $X_b = \varnothing$, and hence

$$M^0 = \| F_{\mu,c} - f \| \geqslant M.$$

Examples for Hölder sets are of course differentiable sets with bounded (nonzero) first derivatives. Condition (Z) can also be satisfied by assuming sufficient smoothness of the functions $B_a \in S$, $|a| = 1$; however in applications it is desirable to keep the set $S \subset \mathbf{C}(X)$ fairly simple to control the condition $\mu \geq S/\lambda$.

A straightforward choice of course is $S = [B]$ where $0 \neq B \in \mathbf{C}(X)$ is some approximation to the function $f \in R(X)$, see Williams [7]. Then we have the following simple result.

LEMMA. *Let $S = [B]$, $0 \neq B \in \mathbf{C}(I)$ with a finite number of zeros in $I \subset \mathbf{R}$, and suppose B has Taylor expansions around each zero. Then condition (Z) is satisfied with some number $\zeta < \infty$.*

But of course condition (Z) is satisfied by more general functions like $x \cdot \sin(\pi/x) \in C(I)$ or $|x| \in C(I)$, $I = [-1, +1]$.

Finally we want to consider the most important application using polynomial rationals. Clearly if $S \subset C(X)$ consists of polynomials with degree $\leq d$ condition (Z) is satisfied for $\xi = d$, and if T consists of polynomials it is a Hölder set with constant $\lambda = 1$, thus we have $M^0 = M$ for $\mu \geq d$. In the one-dimensional case it is easy to show the following sharper result.

THEOREM. *Let $S = [B]$, $0 \neq B \in C(I)$ satisfy condition (Z) with some number $\zeta < \infty$ and suppose B has no zeros on the boundary δI. Let $T \subset C(I)$ consist of polynomials. Then we have $M^0 = M$ for $\mu \geq \zeta/2$.*

For a proof note that if $L \in T$ has a zero at $x \in \operatorname{int}(I)$ it is at least of degree two. B being nonzero on δI can be satisfied by appropriate change of the interval I; and we note that no assumption is made on the degree of the polynomials in T. Thus the result can be generalized to sets T consisting of functions with Taylor expansions around the zeros of B.

The above result compares with Theorem 3.1 in [6], however in the case of interpolation approximation as considered by Taylor and Williams their result can immediately be improved in the sense that their condition (ii) in Theorem 3.1 or our condition "B has no zeros on the boundary δI" is not needed.

THEOREM. *Let $S = [B]$, $0 \neq B \in \mathbf{C}(I)$ satisfy condition (Z) with some number $\xi < \infty$ and let $T \subset \mathbf{C}(I)$ consist of polynomials. Furthermore let $\mu \geq \xi/2$. Then we have $M^0 = M$ for all functions $f \in R(I)$ of the form $f = B \cdot h$ with $h(x) > 0$ $\forall x \in I$.*

Proof. W.l.o.g. let $I = [0, 1]$, and in view of the previous theorem assume $B(0) = 0$. Let $c \in P^0$ such that $\|F_{a,c} - f\| = M^0$, and let $\epsilon > 0$ such that $L_b(x) \neq 0 \quad \forall x \in [-\epsilon, 0)$. Such an $\epsilon > 0$ exists since $\|b\| = 1$ and the polynomials in T are of finite degree. On $[-\epsilon, 0)$ we define B in such a way that $B(x) \neq 0$, $\operatorname{sgn}(B(x)) = \operatorname{sgn}(L_b(x)) \quad \forall x \in [-\epsilon, 0)$, and:

$$\lim_{x \uparrow 0} B(x)/L_b(x) = 0.$$

Put $h(x) = 1/|L_b(x)|$, $f(x) = B(x) \cdot h(x)$, $x \in [-\epsilon, 0)$. Note that $f \in R(I_\epsilon)$ though h has a pole at $x = 0$, and clearly we have

$$\|F_{\mu,c} - f\|_{I_\epsilon} \overset{!}{=} \|F_{\mu,c} - f\|_I .$$

These rather strange continuations of B and h are necessary to include the case $M^0 = 0$, and note that $S = [B] \subset \mathbf{C}(I_\epsilon)$, $I_\epsilon = [-\epsilon, 1]$ satisfies (Z) with the same number $\xi < \infty$. Thus we get from the last equation:

$$M_I^0 \geq M_{I_\epsilon}^0 ,$$

where the subscripts denote the intervals considered. Hence with the previous theorem applied for the interval I_ϵ we conclude:

$$M_I^0 \geq M_{I_\epsilon}^0 \overset{!}{=} M_{I_\epsilon} \geq M_I .$$

We wish to remark that all results in this chapter four in fact, ensure the existence of a best approximation $F_{\mu,c} \in V(N, \mu)$ which is a somewhat stronger result than just the condition $M^0 = M$. If furthermore the sets $V(N, \mu)$ are well chosen, i.e., chosen in such a way that the best approximation is non-trivial, then—with the exception of the last theorem—we can show that any minimal sequence $(F_{\mu,c_j}) \subset V(N, \mu)$ converges to a best approximation, for details see [5].

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